

# HELICAL CR STRUCTURES AND SUB-RIEMANNIAN GEODESICS

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**ABSTRACT.** A helical CR structure is a decomposition of a real Euclidean space into an even-dimensional horizontal subspace and its orthogonal vertical complement, together with an almost complex structure on the horizontal space and a marked vector in the vertical space. We prove an equivalence between such structures and step two Carnot groups equipped with a distinguished normal geodesic, and also between such structures and smooth real curves whose derivatives have constant Euclidean norm. As a consequence, we relate step two Carnot groups equipped with sub-Riemannian geodesics with this family of curves. The restriction to the unit circle of certain planar homogeneous polynomial mappings gives an instructive class of examples. We describe these examples in detail.

## 1. INTRODUCTION

In this paper we identify a connection between CR geometry and sub-Riemannian geometry. We introduce helical CR structures to relate the structure of geodesics in step two Carnot groups with a certain family of smooth real curves.

**Definition 1.1.** A helical CR structure  $\mathcal{C}$  of type  $(2n, p)$  on a Euclidean space  $\mathbb{R}^d = \mathbb{R}^{2n+p}$  is an orthogonal decomposition

$$(1.1) \quad \mathbb{R}^d = \mathbb{R}^{2n} \oplus \mathbb{R}^p$$

together with an invertible skew-symmetric operator  $A$  on  $\mathbb{R}^{2n}$  and a vector  $w \in \mathbb{R}^p$ . The *horizontal space* of  $\mathcal{C}$  is the subspace  $H = \mathbb{R}^{2n}$ , while  $w$  is the *vertical direction*.

A *marked helical CR structure* is a triple  $(\mathcal{C}, v, u_0)$ , where  $\mathcal{C}$  is a helical CR structure,  $v \in \mathbb{R}^{2n}$ , and  $u_0 = v_0 \oplus w_0$  is a vector in  $\mathbb{R}^d = \mathbb{R}^{2n} \oplus \mathbb{R}^p$ . In this case, we also write  $u = v \oplus w$ , where  $w$  is the vector in  $\mathbb{R}^p$  from the previous paragraph.

We say that  $\mathcal{C}$  is (a) *horizontally trivial* if  $n = 0$ , (b) *vertically trivial* if  $p = 0$ , and (c) *completely nontrivial* if  $n > 0$  and  $p > 0$ .

A *Carnot group* is a nilpotent stratified Lie group. Such groups are naturally equipped with so-called *Carnot-Carathéodory metrics*, which are singular metrics (from the perspective of Riemannian geometry). Carnot groups figure prominently as examples in the theory of analysis in metric measure spaces. They also arise as local tangent space models for sub-Riemannian manifolds. Geodesics in sub-Riemannian manifolds yield solutions to various path planning problems in control theory and related applications. See [5] and [16] for more information.

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In section 2 we make precise the equivalence between completely nontrivial helical CR structures and the geometry of step two Carnot groups and their (normal) sub-Riemannian geodesics. The principal results of this section are Proposition 2.1 and Theorem 2.2. The proof of Theorem 2.2 is a straightforward application of the method of bicharacteristics to solve the sub-Riemannian Hamiltonian equations. The explicit form of geodesics in step two Carnot groups dates back to work of Gaveau [12] and Brockett [1], [2]; see Montgomery's book [16] for a very readable summary. However, the precise geometric characterizations of such groups and their geodesics given in Proposition 2.1 and Theorem 2.2 enable us to link their study with an ostensibly very different subject, namely, the study of a family of smooth real curves motivated by homogeneous polynomial mappings between balls and spheres. The first author has made considerable use of such polynomial mappings in complex Euclidean spaces. See [7], [8, Chapter 5], [9], [10]. The notion of a helical CR structure turns out to be equivalent to real curves which are the analogues of such mappings. We thus obtain an elementary differential geometric characterization of such structures and in turn of step two Carnot groups and their geodesics.

More precisely, let us consider the class  $\mathcal{Q}_1$  consisting of smooth curves taking values in a real Euclidean space all of whose derivatives have constant norm, and the subset  $\mathcal{Q}_0$  consisting of curves which have constant norm themselves (i.e., whose image lies in a sphere centered at the origin). In Theorem 3.7 we show that each  $\gamma \in \mathcal{Q}_0$  determines a canonical decomposition  $\mathbb{R}^d = \mathbb{R}^{2n} \oplus \mathbb{R}^p$  and an identification of its *horizontal space*  $\mathbb{R}^{2n}$  with  $\mathbb{C}^n$ . The appearance of a complex structure on the horizontal space is an intriguing phenomenon which is by no means evident from the original definition of the family  $\mathcal{Q}_0$ . It links the study of such curves with the notion of helical CR structure. Indeed, each such curve is of the form  $\gamma(s) = \exp(As)v \oplus w$ , where  $A$  is skew-symmetric and invertible. From a geometric perspective, such curves can be viewed as generalized helices; from a different point of view, the induced helical CR structure naturally determines a set of planes (two-dimensional subspaces of the horizontal space) into which the projection of the curve is a circle. Examples include classical helices in  $\mathbb{R}^3$  as well as the skew-line on the torus  $\mathbb{S}^1 \times \mathbb{S}^1$ . Theorem 3.9 formalizes these observations and summarizes the relationship between helical CR structures and the classes  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$ .

By combining Theorems 2.2 and 3.9 we obtain the following differential geometric characterization of step two Carnot groups of contact type and their geodesics. Theorem 1.2 is restated and proved later in the paper as Corollary 3.10(b).

**Theorem 1.2.** *Each nonaffine curve  $\mu \in \mathcal{Q}_1$  contained in a hyperplane of  $\mathbb{R}^d$  determines a unique step two stratified Lie algebra of contact type on a subspace of  $\mathbb{R}^d$  together with the germ of a normal geodesic  $c$  for the induced Carnot-Carathéodory metric on the associated Lie group. Conversely, each such Lie algebra and geodesic determine a curve in  $\mathcal{Q}_1$ . The horizontal projections of  $\mu$  and  $c$  coincide.*

The simplest example of a completely nontrivial helical CR structure arises from the identification of  $\mathbb{R}^3$  with  $\mathbb{C} \times \mathbb{R}$ ; the skew-symmetric  $A$  is the standard almost complex structure matrix

$$(1.2) \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the vertical direction  $w$  is  $(0, 0, 1)$ . The associated curve in  $\mathcal{Q}_1$  is a right cylindrical helix, while the associated step two Carnot group is the Heisenberg group  $\mathbb{H}^1$ . The sub-Riemannian geodesics in  $\mathbb{H}^1$  are certain nonlinear helices whose horizontal projections are circles in the horizontal space  $\mathbb{R}^2 = \mathbb{C}$ . See Example 2.3.

From the point of view of CR geometry, the unit sphere in  $\mathbb{C}^n$  is a CR manifold of hypersurface type. Its sub-Riemannian geometry is modelled on the Heisenberg group. Theorem 1.2 refers to step two Carnot groups of contact type, i.e., with one missing direction. In Theorem 4.4 we extend the correspondence in Theorem 1.2 to relate arbitrary step two Carnot groups with  $p$ -dimensional center, equipped with  $p$  distinguished normal geodesics, with  $p$ -tuples of elements in  $\mathcal{Q}_1$  with common horizontal space but linearly independent vertical directions. More precisely, for  $1 \leq j \leq p$  let  $\mu_j$  denote elements of  $\mathcal{Q}_1$  taking values in  $\mathbb{R}^d$  and whose horizontal spaces coincide. When their vertical directions are linearly independent we naturally obtain a step two Carnot group of type  $(2n, p)$ . Conversely, given any step two Carnot group, the structure matrices for its Lie algebra together with a basis for its vertical space naturally determine a  $p$ -tuple of such curves.

In section 3.3 we carefully discuss the specific  $\mathcal{Q}_0$  curves which arose from consideration of group-invariant proper mappings. For these curves the corresponding skew-symmetric matrices are bidiagonal. We compute their spectra. In Remark 4.2 we mention a recent paper concerning bidiagonalization of skew-symmetric matrices.

We mention several directions for future work. One may consider the analogous situation where the starting point is certain homogeneous CR mappings between odd-dimensional complex spheres, and try to relate such mappings with harmonic maps between Carnot groups. See [6] for a study of such maps with Heisenberg group target. In the setting of this paper, groups of step two arise because the unit sphere in  $\mathbb{C}^n$  is strongly pseudoconvex. By considering CR maps between CR manifolds with degenerate Levi forms, it might be possible to consider nilpotent groups of higher step. See [17] for the first decisive use of groups of higher step to prove regularity results for differential operators arising in CR geometry. The connection which we identify between CR and sub-Riemannian geometry may also be of interest in the higher codimension setting. See [4] for the construction of partial normal forms for the defining equations of CR manifolds of finite type in higher codimension. The finite type conditions in [4] arise via iterated commutators of complex vector fields and their conjugates. It is also natural to study smoothly varying helical CR structures on the tangent spaces of a manifold.

## 2. STEP TWO CARNOT GROUPS AND HELICAL CR STRUCTURES

In this section we establish the relationship between helical CR structures and the sub-Riemannian geometry of step two Carnot groups and their geodesics. In particular, we prove Proposition 2.1 and Theorem 2.2.

**2.1. Stratified Lie algebras and Lie groups.** Let  $\mathfrak{g}$  be a Lie algebra. We assume that  $\mathfrak{g}$  admits a direct sum decomposition  $\mathfrak{g} = \mathbf{v}_1 \oplus \cdots \oplus \mathbf{v}_s$ . We set  $\mathbf{v}_i = 0$  for  $i > s$ . We say that  $\mathfrak{g}$  is *stratified* if

$$[\mathbf{v}_i, \mathbf{v}_j] = \mathbf{v}_{i+j}$$

for all  $i, j \geq 1$ . The integer  $s$  is called the *step* of the graded or stratified structure.

Let  $\mathbb{G}$  be the Lie group corresponding to  $\mathfrak{g}$ . We identify  $\mathfrak{g}$  with the left-invariant vector fields on  $\mathbb{G}$ . We say that  $\mathbb{G}$  is *graded* or *stratified* if its Lie algebra enjoys the corresponding property. Stratified Lie groups are also known as *Carnot groups*.

Let  $\mathbb{G}$  be a step two Carnot group with Lie algebra  $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$ . We write  $m = \dim \mathfrak{v}_1$  and  $p = \dim \mathfrak{v}_2$ . Directions in  $\mathfrak{v}_1$  are called *horizontal*, while directions in  $\mathfrak{v}_2$  are called *extra* (or *vertical*) directions. The relation  $[\mathfrak{v}_1, \mathfrak{v}_1] = \mathfrak{v}_2$  implies that  $1 \leq p \leq \binom{m}{2}$ . We say that  $\mathfrak{g}$  or  $\mathbb{G}$  is of *contact type* if  $p = 1$ . Some authors (see [16]) reserve this terminology for a stronger bracket-generating condition.

Let  $X_1, \dots, X_m$  be a basis for  $\mathfrak{v}_1$  and let  $T_1, \dots, T_p$  be a basis for  $\mathfrak{v}_2$ . For each  $i, j$  there are constants  $c_{ij}^\alpha$  such that

$$[X_i, X_j] = \sum_{\alpha=1}^p c_{ij}^\alpha T_\alpha.$$

The skew-symmetric matrices  $C^\alpha = (c_{ij}^\alpha)$  are called the *structure matrices* of  $\mathbb{G}$ .

The exponential map is a global diffeomorphism from  $\mathfrak{g}$  to  $\mathbb{G}$ . We identify  $\mathfrak{g}$  with  $\mathbb{R}^{m+p}$ . In the corresponding *exponential coordinates*  $(x_1, \dots, x_m; t_1, \dots, t_p) \in \mathbb{R}^m \oplus \mathbb{R}^p$  we write

$$X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^m \sum_{\alpha=1}^p c_{ij}^\alpha x_j \frac{\partial}{\partial t_\alpha}$$

for  $i = 1, \dots, m$ , and  $T_\alpha = \frac{\partial}{\partial t_\alpha}$  for  $\alpha = 1, \dots, p$ .

We consider briefly two extreme cases. When  $p = 1$  and  $m = 2n$  we obtain the Heisenberg group  $\mathbb{H}^n$ ; this is the unique Carnot group of dimension  $2n + 1$  with one vertical direction. When  $p = \binom{m}{2}$  we obtain the so-called *free* two step nilpotent Lie group  $\mathbb{N}_{m,2}$ . In this case, the number  $p$  of vertical directions is maximal for a given number of horizontal directions  $m$ .

**Proposition 2.1.** *A completely nontrivial helical CR structure on  $\mathbb{R}^d$  uniquely determines and is uniquely determined by a step two stratified Lie algebra of contact type on a subspace of  $\mathbb{R}^d$ .*

*Proof.* Let  $\mathcal{C}$  be a completely nontrivial helical CR structure on  $\mathbb{R}^d$  with  $2n, p, A = (a_{ij}), w$  as in Definition 1.1. Define vector fields  $X_1, \dots, X_{2n}, T_w$  on  $\mathbb{R}^{2n+p}$  by

$$T_w = \sum_{\alpha=1}^p w_\alpha \frac{\partial}{\partial t_\alpha}$$

and

$$(2.3) \quad X_i = \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{j=1}^{2n} a_{ij} x_j T_w.$$

Thus  $T_w$  is the constant vector field in  $\mathbb{R}^{2n+p}$  pointing in the direction of  $w = (w_1, \dots, w_p)$ . The commutation relations  $[X_i, X_j] = a_{ij} T_w$  for  $i, j = 1, \dots, 2n$  and  $[X_i, T_w] = 0$  for  $i = 1, \dots, 2n$  follow easily from (2.3). They equip the vector subspace  $\mathbb{R}^{2n} \oplus \text{span}(T_w)$  with the structure of a step two stratified Lie algebra of contact type.

Conversely, assume that  $\mathfrak{g} = \mathfrak{v}_1 \oplus \mathfrak{v}_2$  is a subspace of  $\mathbb{R}^d$  equipped with the structure of a step two stratified Lie algebra of contact type. Let  $B = (b_{ij})$  be the structure matrix for  $\mathfrak{g}$ , defined by the commutation relations  $[X_i, X_j] = b_{ij} T$  where  $X_1, \dots, X_m$  is a basis for the first layer  $\mathfrak{v}_1$  and  $T$  generates the second layer

$\mathbf{v}_2$ . Let  $w$  span the one-dimensional space  $\exp(T)$ . The skew-symmetric matrix  $B$  need not be invertible, and hence cannot necessarily be used in Definition 1.1. We obtain an invertible skew-symmetric  $A$  by restricting  $B$  to the orthocomplement of its null space. This space is even dimensional because nonzero eigenvalues come in conjugate pairs; thus  $A$  is a  $(2n) \times (2n)$  matrix for some  $n$ . This information now determines an orthogonal decomposition  $\mathbb{R}^d = \mathbb{R}^{2n} \oplus \mathbb{R}^p$ , an appropriate invertible skew-symmetric  $A$ , and  $w \in \mathbb{R}^p$ , i.e., a helical CR structure  $\mathcal{C}$ . Since  $w \neq 0$ ,  $\mathcal{C}$  is vertically nontrivial. Since  $\mathbf{g}$  is of step two,  $\mathcal{C}$  is horizontally nontrivial.  $\square$

**2.2. Carnot-Carathéodory metric and sub-Riemannian geodesics.** Let  $\mathbb{G}$  be a step two Carnot group. We introduce a sub-Riemannian metric  $g_0$  as an inner product on the horizontal distribution  $\mathbf{v}_1$  by declaring the vector fields  $X_1, \dots, X_m$  to be orthonormal. Such a metric defines a global distance function on  $\mathbb{G}$  by the formula

$$(2.4) \quad d_0(P, Q) = \inf \text{length}_{g_0}(\gamma),$$

where the infimum is taken over all smooth curves  $\gamma : [a, b] \rightarrow \mathbb{G}$  joining  $P$  to  $Q$  whose tangent vector  $\gamma'(s)$  lies in the subspace of  $T_{\gamma(s)}\mathbb{G}$  spanned by the vector fields  $X_1, \dots, X_m$ . The bracket-generating condition  $[\mathbf{v}_1, \mathbf{v}_1] = \mathbf{v}_2$  guarantees that  $d_0$  is a metric. See [3] or [16]. A curve  $c$  that realizes the infimum in (2.4) is called a *sub-Riemannian geodesic* joining  $P$  to  $Q$ . Existence and regularity theory for sub-Riemannian geodesics in Carnot groups is the focus of the papers [18] and [19].

A curve  $c$  in  $\mathbb{G}$  is called a *horizontal lift* of a curve  $\gamma : [a, b] \rightarrow \exp \mathbf{v}_1$  if  $\pi \circ c = \gamma$ , where  $\pi : \mathbb{G} \rightarrow \exp \mathbf{v}_1$  denotes the projection  $\pi(x_1, \dots, x_m; t_1, \dots, t_p) = (x_1, \dots, x_m)$ . By the general theory of fiber bundles, for each curve  $\gamma$  as above and each  $P \in \pi^{-1}(\gamma(a))$ , there exists a unique horizontal lift  $c$  of  $\gamma$  with  $c(a) = P$ .

A fundamental feature of sub-Riemannian geometry which distinguishes it from its Riemannian counterpart is the potential existence of (strictly) abnormal geodesics. A sub-Riemannian geodesic  $c$  is called *normal* if it is the projection to  $\mathbb{G}$  of a bicharacteristic curve in  $T^*\mathbb{G}$ , i.e., a solution to the sub-Riemannian Hamiltonian system. The book [16] carefully summarizes the delicate issues related to the regularity of sub-Riemannian geodesics and especially to the existence of abnormal minimizers. In the proof of the following theorem we make use of the Hamiltonian formalism, which explains the restriction to normal geodesics. Normality of geodesics in step two Carnot groups is studied in the paper [15].

**Theorem 2.2.** *A completely nontrivial marked helical CR structure on  $\mathbb{R}^d$  uniquely determines and is uniquely determined by a pair  $(S, c)$ , where  $S$  is a step two stratified Lie algebra of contact type on a subspace of  $\mathbb{R}^d$  and  $c$  is the germ of a normal geodesic for the sub-Riemannian (Carnot-Carathéodory) metric on the associated Carnot group.*

*Proof.* Let  $(\mathcal{C}, v, u_0)$  be a completely nontrivial marked helical CR structure on  $\mathbb{R}^d$ . As in the proof of part (a) we introduce a step two stratified Lie algebra of contact type on a subspace of  $\mathbb{R}^d$ . Let  $X_i$  be the vector fields defined in (2.3). Let  $g_0$  be the sub-Riemannian metric on  $\mathbb{G}$  for which the vector fields  $X_i$  are an orthonormal basis for the horizontal space. Normal geodesics in  $\mathbb{G}$  can be identified by the method of bicharacteristics, projecting to  $\mathbb{G}$  the solutions of Hamilton's equations for the sub-Riemannian Hamiltonian. We use coordinates  $(x_1, \dots, x_{2n}, t)$  for points in  $\mathbb{G}$ ,

where  $t$  denotes the coefficient of  $w$  in the vertical direction. The Hamiltonian is

$$H(x, t; \xi, \tau) = \frac{1}{2} \sum_i \zeta_i^2,$$

where  $\zeta_i = \xi_i + \frac{1}{2} \sum_{j=1}^{2n} a_{ij} x_j \tau$  is dual to  $X_i$ . The bicharacteristics  $(x(s), t(s); \xi(s), \tau(s))$  are solutions to Hamilton's equations  $\dot{x} = \nabla_\xi H$ ,  $\dot{t} = \partial H / \partial \tau$ ,  $\dot{\xi} = -\nabla_x H$ ,  $\dot{\tau} = -\partial H / \partial t$ . Explicitly,

$$\dot{x}_i = \zeta_i, \quad \dot{t} = \frac{1}{2} \sum_j a_{ij} x_j \zeta_i, \quad \dot{\xi}_j = -\frac{1}{2} \sum_i a_{ij} \zeta_i \tau, \quad \dot{\tau} = 0$$

with initial conditions  $x(0) = x_0$ ,  $t(0) = t_0$ ,  $\xi(0) = \xi_0$  and  $\tau(0) = \tau_0$ . Thus  $\tau(s) = \tau_0$  is constant in  $s$ . Hamilton's equations can be written in the compact form

$$\dot{x} = \zeta, \quad \dot{t} = \frac{1}{2} \zeta^T A x, \quad \dot{\xi} = \frac{\tau_0}{2} A \zeta, \quad \zeta = \xi + \frac{\tau_0}{2} A x.$$

Thus  $\dot{\zeta} = \tau_0 A \zeta$  and the curve  $\zeta(s) = \exp(s\tau_0 A) \zeta(0)$  is in  $\mathcal{Q}_0$ . By an affine reparameterization we may assume that either  $\tau_0 = 0$  or  $\tau_0 = 1$ .

We next obtain the following constant coefficient coupled ODE system in  $\mathbb{R}^{2n}$  for the first-layer position and momentum variables  $x, \xi$ :

$$\dot{x} = \frac{\tau_0}{2} A x + \xi, \quad \dot{\xi} = \frac{\tau_0^2}{4} A^2 x + \frac{\tau_0}{2} A \xi,$$

which we easily integrate to find

$$\begin{aligned} x(s) &= \frac{1}{2} (\exp(sA) + I) x_0 + (\exp(sA) - I) A^{-1} \xi_0, \\ \xi(s) &= \frac{1}{4} (\exp(sA) - I) A x_0 + \frac{1}{2} (\exp(sA) + I) \xi_0 \end{aligned}$$

if  $\tau_0 = 1$ , or

$$x(s) = x_0 + s \xi_0, \quad \xi(s) = \xi_0$$

if  $\tau_0 = 0$ . Note that in the case  $\tau_0 = 1$  we may write

$$(2.5) \quad x(s) = \left( \frac{1}{2} x_0 - A^{-1} \xi_0 \right) + \exp(sA) \left( \frac{1}{2} x_0 + A^{-1} \xi_0 \right).$$

In either case we immediately see that  $x(s)$  is the projection to  $\mathbb{R}^{2n}$  of a curve in  $\mathcal{Q}_1$  obtained as the integral of the curve  $\gamma(s) = \exp(sA) v \oplus w$  for  $v = \xi_0 + \frac{1}{2} x_0$ .

Conversely, assume that the pair  $(S, c)$  consists of a subspace  $S$  of  $\mathbb{R}^d$  equipped with a step two stratified Lie algebra of contact type and a normal geodesic  $c : (-\epsilon, \epsilon) \rightarrow (\exp(S), g_0)$ , where  $g_0$  denotes the Carnot-Carathéodory metric on  $\exp(S)$ . As above, we write  $c(s) = (x(s), t(s))$  and denote by  $(\xi(s), \tau(s))$  the corresponding momenta variables. As in the proof of part (a), we associate to this data a completely nontrivial helical CR structure  $\mathcal{C}$  on  $\mathbb{R}^d$ . To identify the marking  $(\mathcal{C}, v, u_0)$ , we set  $u_0 = c(0)$ . The discussion in the previous paragraph shows that the projection of  $c$  in the horizontal space  $\exp \mathbf{v}_1$  takes the form (2.5) for suitable  $x_0$  and  $\xi_0$ . The proof is complete upon setting  $v = \frac{1}{2} A x_0 + \xi_0$ .  $\square$

**Example 2.3.** The simplest step two Carnot group is the Heisenberg group  $\mathbb{H}^1$ , with Lie algebra  $\mathfrak{h}^1 = \mathbf{v}_1 \oplus \mathbf{v}_2 = \text{span}\{X, Y\} \oplus \text{span}\{T\}$  and a single nontrivial bracket relation  $[X, Y] = T$ . Sub-Riemannian geodesics in  $\mathbb{H}^1$  take the form

$$s \mapsto (a + b e^{-is}, c + |b|^2 s - \text{Im}(\bar{a} b e^{-is})),$$

see e.g. [3] or [5]. The case  $b = -a$ ,  $c = 0$  corresponds to geodesics issuing from the identity (origin) in  $\mathbb{H}^1 = \mathbb{R}^3$ :

$$s \mapsto (a - ae^{-is}, |a|^2(s - \sin s)).$$

The associated  $\mathcal{Q}_1$  curves from Theorem 1.2 are right cylindrical helices of the form

$$s \mapsto (a - ae^{-is}, c + s)$$

with  $a \in \mathbb{C}$  and  $c \in \mathbb{R}$ . Note that all geodesics in  $\mathbb{H}^1$  (or more generally, in any step two Carnot group satisfying the strong bracket-generating assumption) are normal.

### 3. A CHARACTERIZATION OF HELICAL CR STRUCTURES

In this section, we show that the notion of helical CR structure can be recovered from an elementary concept in real differential geometry. We will study smooth real Euclidean curves all of whose derivatives have constant norm. Such curves can be thought of as “generalized helices”. We begin by developing some basic properties of such curves. For instance, we show that the G-curvatures (coupling parameters in the generalized Frenet formulas) of such a curve are all constant in time. The main point of the section is that every such curve induces a natural decomposition of its target space into horizontal and vertical directions thereby leading to a helical CR structure. See Theorem 3.7.

**3.1. Curves with derivatives of constant norm.** A *smooth real curve* in  $\mathbb{R}^d$  is a smooth map  $\gamma$  from an interval in  $\mathbb{R}$  to  $\mathbb{R}^d$ . We say that  $\gamma$  *lies in* a set  $S$  if its image is a subset of  $S$ . For  $j \geq 0$ , the  $j$ th order derivative of  $\gamma$  will be written  $D^j\gamma$ . We let  $\langle v, w \rangle$  denote the Euclidean inner product of vectors  $v$  and  $w$  in  $\mathbb{R}^d$ , and  $\|v\|^2$  denote the corresponding squared norm. We use the same notation without regard to the dimension  $d$ . Let  $O(d)$  denote the orthogonal group acting on  $\mathbb{R}^d$ .

**Definition 3.1.** Let  $k \geq 0$ . We denote by  $\mathcal{Q}_k$  the collection of smooth real curves  $\gamma$  such that  $D^j\gamma$  has constant Euclidean norm whenever  $j \geq k$ .

Thus  $\gamma \in \mathcal{Q}_0$  if for  $j \geq 0$  there are constants  $c_j$  such that

$$(3.6) \quad \|D^j\gamma(s)\|^2 = c_j$$

for all  $s$ . Using  $j = 0$  in (3.6) we see that  $\gamma$  lies in a sphere centered at the origin. On the other hand, elements of  $\mathcal{Q}_1$  need not have bounded image.

For any skew-symmetric  $A$  and  $v \in \mathbb{R}^d$ , the curve  $\gamma$  defined by  $\gamma(s) = \exp(As)v$  is in  $\mathcal{Q}_0$ . Indeed, since  $A$  commutes with  $\exp(sA)$  and  $\|\exp(As)v\|^2 = \|v\|^2$  for every  $s$ , we obtain

$$\|D^k\gamma(s)\|^2 = \|\exp(As)A^k v\|^2 = \|A^k v\|^2.$$

Thus  $\|D^k\gamma(\cdot)\|$  is constant. In particular, if  $\gamma$  satisfies the ODE  $D\gamma = A\gamma$  for some skew-symmetric  $A$ , then  $\gamma \in \mathcal{Q}_0$ . More generally, if  $A$  is skew-symmetric on  $\mathbb{R}^d$ ,  $B \in O(d)$ , and  $v \in \mathbb{R}^d$ , then the curve  $\gamma$  defined by  $\gamma(s) = B \exp(As)v$  is in  $\mathcal{Q}_0$ . In fact,  $\gamma(s) = \exp(Cs)\gamma(0)$  where  $C = BAB^{-1}$  is skew-symmetric. Theorem 3.7 provides a converse to these assertions.

**Remark 3.2.** A smooth curve in  $O(d)$  need not be a member of  $\mathcal{Q}_0$ . For example, put  $\gamma(s) = A(s)v$ , where  $v$  is a unit vector in  $\mathbb{R}^2$  and

$$A(s) = \begin{pmatrix} \cos f(s) & -\sin f(s) \\ \sin f(s) & \cos f(s) \end{pmatrix}.$$

Then  $\gamma \in \mathcal{Q}_0$  if and only if  $f$  is affine. In general, membership in  $\mathcal{Q}_0$  is preserved only under affine reparameterization. See Remark 4.1 for a discussion of the issues concerning reparameterization.

We discuss some operations which preserve the class  $\mathcal{Q}_0$ . The ideas here are related to the work in [8] concerning polynomial CR mappings between spheres of different dimensions. Parts (i) and (iii) of Lemma 3.3 show that  $\mathcal{Q}_0$  is closed under orthogonal postcomposition and tensor products. Part (ii) exhibits homotopies joining arbitrary elements of  $\mathcal{Q}_0$  provided the target dimension is increased appropriately.

**Lemma 3.3.** *Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$  and  $\mu : \mathbb{R} \rightarrow \mathbb{R}^k$  be elements of  $\mathcal{Q}_0$ , let  $T \in O(n)$ , and let  $\theta \in \mathbb{R}$ . Then*

- (i)  $T \circ \gamma \in \mathcal{Q}_0$ ,
- (ii) the  $\theta$ -juxtaposition  $J_\theta(\gamma, \mu) = (\cos \theta)\gamma \oplus (\sin \theta)\mu$  is in  $\mathcal{Q}_0$ ,
- (iii) the tensor product  $\gamma \otimes \mu$  is in  $\mathcal{Q}_0$ .

*Proof.* The proofs of (i) and (ii) are simple computations:

$$(3.7) \quad \|D^k(T \circ \gamma)(s)\|^2 = \|T \circ D^k\gamma(s)\|^2 = \|D^k\gamma(s)\|^2 = c_k$$

and

$$\begin{aligned} \|D^k J_\theta(\gamma, \mu)(s)\|^2 &= \|J_\theta(D^k\gamma, D^k\mu)(s)\|^2 = \|(\cos \theta)D^k\gamma(s) \oplus (\sin \theta)D^k\mu(s)\|^2 \\ &= \|(\cos \theta)D^k\gamma(s)\|^2 + \|(\sin \theta)D^k\mu(s)\|^2 = \cos^2 \theta \|D^k\gamma(s)\|^2 + \sin^2 \theta \|D^k\mu(s)\|^2. \end{aligned}$$

The latter expression is independent of  $s$  since  $\gamma, \mu \in \mathcal{Q}_0$ .

Finally, we prove (iii). Let  $f$  and  $g$  be arbitrary curves in Euclidean spaces of possibly different dimensions. By definition

$$(3.8) \quad \langle f \otimes g, u \otimes v \rangle = \langle f, u \rangle \langle g, v \rangle$$

and in particular

$$(3.9) \quad \|f \otimes g\|^2 = \|f\|^2 \|g\|^2.$$

It follows from (3.9) that  $\frac{d}{dt}\|f \otimes g\|^2 = \frac{d}{dt}\|f\|^2 \|g\|^2 = 2\langle f, f' \rangle \|g\|^2 + \|f\|^2 2\langle g, g' \rangle$ . Note that if  $\|h\|^2$  is constant, then

$$(3.10) \quad 0 = \frac{d}{dt}\|h\|^2 = 2\langle h, h' \rangle.$$

Applying (3.10) for both  $h = f$  and  $h = g$ , and using (3.8) and (3.9) yields

$$\begin{aligned} (3.11) \quad \left\| \frac{d}{dt}(f \otimes g) \right\|^2 &= \|f' \otimes g + f \otimes g'\|^2 \\ &= \|f' \otimes g\|^2 + 2\langle f', f \rangle \langle g, g' \rangle + \|f \otimes g'\|^2 = \|f'\|^2 \|g\|^2 + \|f\|^2 \|g'\|^2. \end{aligned}$$

Thus  $f \otimes g$  and its first derivative have constant norm whenever  $f, g \in \mathcal{Q}_0$ . An easy induction on the number of derivatives now yields the conclusion in (iii).  $\square$

The classes  $\mathcal{Q}_0$  or  $\mathcal{Q}_1$  have connections with generalized Frenet frames and higher curvatures. Gluck [13], [14] considered smooth curves in  $\mathbb{R}^n$  whose first  $n$  derivatives are linearly independent at each point. Applying Gram-Schmidt yields an orthonormal frame along the curve; by differentiating this frame field, one obtains higher curvatures (so-called G-curvatures), which depend only upon the inner products  $e_{kl}(s) = \langle D^k\gamma(s), D^l\gamma(s) \rangle$  for  $k, l \leq n$ . The following lemma shows that all such inner products  $e_{kl}(s)$  with  $k, l \geq 1$  are independent of  $s$  for a curve  $\gamma \in \mathcal{Q}_1$ . We will



use Corollary 3.5 in the proof of our characterization theorem (Theorem 3.7). It also implies that all G-curvatures are constant in the time parameter for such curves.

**Lemma 3.4.** *Let  $\gamma \in \mathcal{Q}_0$ . Then all inner products  $e_{kl}(s)$  are independent of  $s$  and  $e_{kl} = 0$  when  $k - l$  is odd. When  $\gamma \in \mathcal{Q}_1$ , the same conclusions hold for all  $k, l \geq 1$ .*

**Corollary 3.5.** *Let  $\gamma \in \mathcal{Q}_0$ . Let  $W_e(s)$  denote the span of the derivatives of  $\gamma$  of even order at  $s$  (including  $\gamma(s)$  itself), and let  $W_o(s)$  denote the span of the derivatives of  $\gamma$  of odd order at  $s$ . Then, for each  $s$ , the spaces  $W_e(s)$  and  $W_o(s)$  are orthogonal.*

*Proof of Lemma 3.4.* Assume (3.6) holds. For each  $k \geq 0$  we have

$$(3.12) \quad 0 = D \left( \|D^k \gamma(s)\|^2 \right) = 2 \langle D^k \gamma(s), D^{k+1} \gamma(s) \rangle.$$

By (3.12) the conclusion holds when  $l = k + 1$ . Differentiating again shows that

$$(3.13) \quad 0 = \|D^{k+1} \gamma(s)\|^2 + \langle D^k \gamma(s), D^{k+2} \gamma(s) \rangle.$$

The squared norm term in (3.13) is a constant, and hence so is the inner product term. Therefore  $e_{kl}$  is a constant when  $l = k + 2$ . By differentiating (3.13) and using induction we obtain the first statement for  $\mathcal{Q}_0$ . By differentiating (3.12) an even number of times and using induction we obtain the second statement for  $\mathcal{Q}_0$ . The statement for  $\mathcal{Q}_1$  follows because  $\gamma \in \mathcal{Q}_1$  implies  $D\gamma \in \mathcal{Q}_0$ .  $\square$

**3.2. A characterization of  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$ .** In Lemma 3.6 we show that each element  $\gamma \in \mathcal{Q}_0$  satisfies a constant coefficient ODE, whose characteristic equation has only purely imaginary solutions. From this we deduce Theorem 3.7, which demonstrates how  $\gamma$  induces a helical CR structure in the target space.

**Lemma 3.6.** *Let  $\gamma \in \mathcal{Q}_0$ . Then there is a positive integer  $k$  and constant scalars  $C_j$  for  $0 \leq j < k$  such that*

$$(3.14) \quad D^{2k} \gamma = \sum_{j=0}^{k-1} C_j D^{2j} \gamma.$$

*Proof.* Since the target space is finite dimensional, for each  $s$  there is a smallest  $k$  so that a non-trivial dependence among the even derivatives of  $\gamma$  holds in a neighborhood of  $s$ :

$$(3.15) \quad D^{2k} \gamma(s) = \sum_{j=0}^{k-1} C_j(s) D^{2j} \gamma(s).$$

Observe that the coefficient functions  $C_j$  in (3.15) are smooth. Differentiating (3.15) gives

$$(3.16) \quad D^{2k+1} \gamma(s) = \sum_{j=0}^{k-1} C_j(s) D^{2j+1} \gamma(s) + \sum_{j=0}^{k-1} C'_j(s) D^{2j} \gamma(s).$$

The spaces  $W_e(s)$  and  $W_o(s)$  are orthogonal by Corollary 3.5. Orthogonality of the appropriate terms in (3.16) forces

$$(3.17) \quad 0 = \sum_{j=0}^{k-1} C'_j(s) D^{2j} \gamma(s).$$

Since (3.17) gives a linear dependence among the even derivatives for a smaller value of  $k$ , and  $k$  was chosen minimally, relation (3.17) must be trivial. Thus  $C'_j(s)$  vanishes for each  $j$ ,  $C_j$  is constant for each  $j$ , and (3.14) holds.  $\square$

**Theorem 3.7.** *Let  $\gamma$  be a curve in  $\mathcal{Q}_0$  taking values in  $\mathbb{R}^d$ . Then  $\gamma$  canonically determines the following data:*

- an orthogonal decomposition of the target space  $\mathbb{R}^d = \mathbb{R}^{2n} \oplus \mathbb{R}^p$ ,
- an invertible skew-symmetric linear map  $A$  on  $\mathbb{R}^{2n}$ ,
- a complex structure  $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ ,
- vectors  $v \in \mathbb{R}^{2n}$  and  $w \in \mathbb{R}^p$ , for which

$$(3.18) \quad \gamma(s) = \exp(As)v \oplus w.$$

The curve  $\gamma$  is constant if and only if  $n = 0$  and  $d = p$ . It lies in no hyperplane if and only if  $d = 2n$  and  $p = 0$ .

**Corollary 3.8.** *Let  $\mu \in \mathcal{Q}_1$ . With  $A, v, w$  as above, there are  $v_0$  and  $w_0$  such that*

$$(3.19) \quad \mu(s) = ((\exp(As) - I)A^{-1}v + v_0) \oplus (ws + w_0).$$

*Proof.* Put  $\gamma = D\mu$ . Then (3.19) follows by integrating (3.18).  $\square$

In Theorem 3.7 and Corollary 3.8 we allow the possibilities that  $n = 0$  or  $p = 0$ . Note that  $n = 0$  if and only if the curve  $\gamma \in \mathcal{Q}_0$  is constant, or equivalently if its integral  $\mu \in \mathcal{Q}_1$  is affine. The copy of  $\mathbb{R}^{2n}$  is the *horizontal subspace* defined by  $\gamma$ . The constant vector  $w \in \mathbb{R}^p$  is the direction of the vertical component of  $\mu$ .

To prove Theorem 3.7 we will show that  $\gamma$  solves a constant coefficient ODE. Hence there are constants  $\lambda_j \in \mathbb{C}$  and (not identically zero) vector-valued complex polynomials  $q_j$  such that

$$(3.20) \quad \gamma(s) = \sum_{j=1}^K q_j(s) e^{\lambda_j s}.$$

We will show that each  $\lambda_j$  is purely imaginary and then that each polynomial  $q_j$  is constant. These statements have the following consequence. By (3.14) there is a polynomial  $p$  such that  $p(D)\gamma = 0$ . The numbers  $\lambda_j$  in (3.20) are the roots of  $p$ . These roots are distinct if and only if each  $q_j$  is constant.

Conversely, as previously observed, any curve of the form (3.18) is in  $\mathcal{Q}_0$  and any curve of the form (3.19) is in  $\mathcal{Q}_1$ . As a consequence, we characterize step two Carnot groups in terms of  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$  curves. See Theorem 3.9 and Corollary 3.10.

*Proof of Theorem 3.7.* By the preceding discussion, we may assume that  $\gamma$  is non-constant. Formula (3.20) holds because  $\gamma \in \mathcal{Q}_0$ , and  $K \geq 1$  because  $\gamma$  is nonconstant. We first claim that all the  $\lambda_j$  in (3.20) must be purely imaginary. Write  $\lambda_j = \xi_j + i\eta_j$  and assume that  $|\xi_j| \leq |\xi_1|$  for all  $j$ . Then  $c_0 = \|\gamma(s)\|^2 = e^{2|\xi_1|s} \Phi(s)$  with  $\limsup_{s \rightarrow \infty} \Phi(s) > 0$ . Thus  $0 = \xi_1$  and hence  $0 = \xi_j$  for all  $j$ .

Next we claim that each polynomial  $q_j$  is a constant. In other words, there are no repeated roots (no resonances). The proof is a similar asymptotic argument. Suppose that the maximal degree of the  $q_j$  is  $d$ . Then  $c_0 = |s|^{2d} \Psi(s)$  with  $\limsup_{s \rightarrow \infty} \Psi(s) > 0$ . Thus  $d = 0$  and each polynomial  $q_j$  is a (nonzero) constant.

To this point we have shown that there is a polynomial  $p$  with distinct, purely imaginary zeros such that  $p(D)\gamma = 0$ . We write  $q_j = x_j + iy_j$  for vectors  $x_j$  and  $y_j$

and put  $\lambda_j = i\eta_j$ . Since  $\gamma$  is real, (3.20) yields

$$(3.21) \quad \gamma(s) = \sum_{j=1}^K x_j \cos(\eta_j s) - y_j \sin(\eta_j s).$$

The coefficients  $x_j$  and  $y_j$  in (3.21) are elements of  $\mathbb{R}^d$ .

Recall that the  $\eta_j$  are distinct. Let  $\eta_1, \dots, \eta_n$  denote the nonzero values and let  $\eta_{n+1} = 0$  if necessary. Put  $v = \sum_{j=1}^n x_j$  and put  $w = \sum_{j>n} x_j$  in (3.21). Since  $\gamma$  is orthogonal to  $D\gamma$ , we can rewrite (3.21) as

$$(3.22) \quad \gamma(s) = \left( \sum_{j=1}^n x_j \cos(\eta_j s) - y_j \sin(\eta_j s) \right) \oplus w$$

where  $\gamma(0) = v \oplus w$ .

We claim that the set of vectors  $x_1, y_1, \dots, x_n, y_n$  are linearly independent. Given the claim their span is a canonical copy of  $\mathbb{R}^{2n}$  in  $\mathbb{R}^d$ . Thus, given  $\gamma \in \mathcal{Q}_0$ , we obtain via (3.22) a canonical subspace  $\mathbb{R}^{2n}$  of  $\mathbb{R}^d$  and its orthogonal complement  $\mathbb{R}^p$ .

The linear independence and the existence of the skew-symmetric  $A$  follow from the following reasoning. Let  $J$  be the usual complex structure matrix in (1.2) and let  $A$  denote the direct sum of the blocks  $\eta_j J$  for  $j \leq n$ . Then (for  $n \neq 0$ ) the matrix  $A$  is invertible and  $\exp(As)$  is the direct sum of blocks

$$(3.23) \quad \begin{pmatrix} \cos(\eta_j s) & -\sin(\eta_j s) \\ \sin(\eta_j s) & \cos(\eta_j s) \end{pmatrix}.$$

By combining (3.22) and (3.23) one obtains (3.18) for vectors  $v$  and  $w$ . If  $n > 0$  then  $v \neq 0$ .

The linear transformation  $A$  has  $2n$  distinct nonzero eigenvalues  $\pm i\eta_1, \dots, \pm i\eta_n$ . Thus for any nonzero vector  $v$ , the vectors  $A^j v$  for  $0 \leq j \leq 2n-1$  are linearly independent. We conclude that  $\gamma$  lies in no hyperplane if and only if  $p = 0$ .  $\square$

Theorem 3.9 and Corollary 3.10 provide the decisive relationship among step two Carnot groups and their geodesics, helical CR structures, and curves in  $\mathcal{Q}_0$  and  $\mathcal{Q}_1$ . Part (b) of Corollary 3.10 restates Theorem 1.2 from the introduction.

**Theorem 3.9.** (a) Each curve  $\gamma \in \mathcal{Q}_0$  determines a helical CR structure on its target space whose horizontal space coincides with that of  $\gamma$ . Conversely, each helical CR structure with horizontal space  $H$  determines a family of curves  $\gamma \in \mathcal{Q}_0$ , parameterized by  $\pi_H(\gamma(0))$ , where  $\pi_H$  denotes the projection from  $\mathbb{R}^d$  onto  $H$ .

(b) Each curve  $\mu \in \mathcal{Q}_1$  determines a marked helical CR structure on its target space whose horizontal space coincides with that of  $\mu$ . Conversely, each marked helical CR structure determines a curve in  $\mathcal{Q}_1$ .

**Corollary 3.10.** (a) Each nonconstant curve in  $\mathcal{Q}_0$  contained in a hyperplane of  $\mathbb{R}^d$  determines a unique step two stratified Lie algebra of contact type on a subspace of  $\mathbb{R}^d$ . Conversely, each such Lie algebra with horizontal space  $H$  determines a family of curves  $\gamma \in \mathcal{Q}_0$ , parameterized by  $\pi_H(\gamma(0))$ .

(b) Each nonaffine curve  $\mu \in \mathcal{Q}_1$  contained in a hyperplane of  $\mathbb{R}^d$  determines a unique step two stratified Lie algebra of contact type on a subspace of  $\mathbb{R}^d$  together with the germ of a normal geodesic  $c$  for the induced Carnot-Carathéodory metric on the associated Lie group. Conversely, each such Lie algebra and geodesic determine a curve in  $\mathcal{Q}_1$ . The horizontal projections of  $\mu$  and  $c$  coincide.

Theorem 3.7 has additional geometric consequences. Let  $\gamma$  be a curve in  $\mathcal{Q}_0$  which for convenience we assume to be vertically trivial. By Theorem 3.7 we have  $\gamma(s) = \exp(As)v$  for some skew symmetric  $A$  and  $v \in \mathbb{R}^{2n}$ . Since the eigenvalues of  $A$  are distinct,  $\mathbb{R}^{2n}$  is the direct sum of two-dimensional subspaces which by (1.2) and (3.23) we may identify with  $\mathbb{C}$ . The projection of  $\gamma$  into each of these particular planes is a circle. Of course  $\gamma$  is in general not a circle, because its image does not lie in a plane. A well-known example is the *skew-line*  $\phi : \mathbb{R} \rightarrow \mathbb{S}^1 \times \mathbb{S}^1$  defined by  $\phi(s) = (e^{is}, e^{i\alpha s})$  for irrational  $\alpha$ , which embeds  $\mathbb{R}$  injectively in  $\mathbb{S}^1 \times \mathbb{S}^1$ . Let  $\gamma$  be an element of  $\mathcal{Q}_0$  which maps into no hyperplane in  $\mathbb{R}^{2n}$ . Theorem 3.7 allows us to identify  $\mathbb{R}^{2n}$  with  $\mathbb{C}^n$  and to think of  $\gamma$  as being defined by

$$(3.24) \quad \gamma(s) = (\zeta_1 e^{i\alpha_1 s}, \dots, \zeta_n e^{i\alpha_n s}).$$

From (3.24) we obtain a simple criterion for injectivity of  $\mathcal{Q}_1$  curves.

**Corollary 3.11.** *A curve  $\gamma \in \mathcal{Q}_1$  fails to be injective if and only if each  $\alpha_j$  in (3.24) is a rational multiple of  $\alpha_1$ . In this case  $\gamma$  traces its image countably many times.*

*Proof.* Injectivity fails if and only if there are distinct real  $s$  and  $u$  such that  $e^{i\alpha_j u} = e^{i\alpha_j s}$  for all  $j$ , equivalently, if and only if  $\alpha_j(u - s) = 2\pi k_j$  for some integers  $k_j$ .  $\square$

**3.3. Homogeneous elements of  $\mathcal{Q}_0$ .** In this section we study specific curves in  $\mathcal{Q}_0$  defined by homogeneous polynomial expressions analogous to those in [7]. The study of these examples motivated the present work. We describe the associated helical CR structures and determine the spectra of the relevant skew-symmetric matrices. In Remark 3.15 we briefly discuss the corresponding polynomial mappings in several complex variables.

For  $0 \leq j \leq m$  let  $E_j$  denote the standard  $j$ th basis vector of  $\mathbb{R}^{m+1}$ . We define the curve  $\gamma_m : \mathbb{R} \rightarrow \mathbb{R}^{m+1}$  by the formula

$$(3.25) \quad \gamma_m(s) = \sum_{j=0}^m \sqrt{\binom{m}{j}} \cos^{m-j}(s) \sin^j(s) E_j.$$

We say that a curve  $\gamma$  with values in  $\mathbb{R}^d$  is *homogeneous of degree  $m$*  if

$$(3.26) \quad \gamma(s) = H(\cos s, \sin s)$$

for some  $H \in V_m(2, d)$ , where  $V_m(2, d)$  denotes the collection of homogeneous polynomial maps of degree  $m$  in two real variables with values in  $\mathbb{R}^d$ .

**Proposition 3.12.** *For each  $m$ ,  $\gamma_m$  lies in the unit sphere,  $\gamma_m \in \mathcal{Q}_0$ , and  $\gamma_m$  is homogeneous of degree  $m$ .*

*Proof.* From (3.25) we see that  $\gamma_m$  is homogeneous of degree  $m$  and that  $\gamma_m(0)$  is the unit vector  $E_0$ . We claim that  $\gamma_m$  satisfies the ODE  $D\gamma_m = L_m \gamma_m$  where  $L_m$  is skew-symmetric. It follows that  $\|\gamma_m(s)\| = \|\gamma_m(0)\| = 1$  for all  $s$ . Thus it suffices to prove the claim.

We determine  $L_m$  by using the elementary identity

$$(3.27) \quad \frac{d}{ds} (\cos^a(s) \sin^b(s)) = b \cos^{a+1}(s) \sin^{b-1}(s) - a \cos^{a-1}(s) \sin^{b+1}(s).$$

With respect to the standard basis the matrix entries of  $L_m$  all vanish except on the super- and sub-diagonals. Furthermore each entry on the superdiagonal is negative, and the corresponding entry on the subdiagonal is its additive inverse. It follows that  $L_m$  is skew-symmetric and hence that  $\gamma_m \in \mathcal{Q}_0$ .  $\square$

We give a second explanation for why  $\gamma_m \in \mathcal{Q}_0$ . By part (iii) of Lemma 3.3, the tensor product of elements in  $\mathcal{Q}_0$  is also in  $\mathcal{Q}_0$ . We can identify  $\gamma_m$  with the  $m$ -fold symmetric tensor product of  $\gamma_1$ , whose image is the unit circle. It is also easy to see why  $\gamma_m$  lies in the unit sphere. By (3.9) we have

$$(3.28) \quad \|\gamma_m(s)\|^2 = \|\gamma_1(s)\|^{2m} = (\cos^2 s + \sin^2 s)^m = 1.$$

Let  $\xi$  be a curve that is homogeneous of degree  $m$ . The monomials  $x^{m-k}y^k$  for  $0 \leq k \leq m$  form a basis for  $V_m(2, 1)$ . The component functions of  $\gamma_m$  therefore span the space of homogeneous polynomials of degree  $m$  in  $\cos s$  and  $\sin s$ . It follows that there a linear mapping  $B$  such that

$$(3.29) \quad \xi(s) = B\gamma_m(s) = B \exp(L_m s) E_0.$$

The mapping  $B$  need not be orthogonal, nor even invertible, even if  $\xi \in \mathcal{Q}_0$ .

**Example 3.13.** Let  $\xi(s) = (\cos^2 s - \sin^2 s, 2 \cos s \sin s, 0)$ . Then  $\xi$  is homogeneous of degree 2, and hence can be written  $B\gamma_2$  for some  $B$ . In this case  $B$  is not invertible, yet  $\xi \in \mathcal{Q}_0$ . The image is the unit circle.

We write matrix representations for the first few  $L_m$  and explicit formulas for the corresponding  $\gamma_m$ . By convention  $\gamma_0$  is the constant 1. We have

$$(3.30) \quad L_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\gamma_1(s) = (\cos s, \sin s),$$

$$L_2 = \begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix},$$

$$(3.31) \quad \gamma_2(s) = (\cos^2 s, \sqrt{2} \cos s \sin s, \sin^2 s),$$

$$L_3 = \begin{pmatrix} 0 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix},$$

$$(3.32) \quad \gamma_3(s) = (\cos^3 s, \sqrt{3} \cos^2 s \sin s, \sqrt{3} \cos s \sin^2 s, \sin^3 s).$$

Since  $L_m$  is skew-symmetric, all its eigenvalues are purely imaginary. In fact, for each  $m$ , the eigenvalues of  $L_m$  are distinct, and hence  $L_m$  is diagonalizable over  $\mathbb{C}$ . We next determine the characteristic polynomial of the  $(m+1) \times (m+1)$  matrix  $L_m$ . We write  $p_m(x)$  for the characteristic polynomial  $\det(L_m - xI)$ .

**Proposition 3.14.** *The characteristic polynomial  $p_m(x)$  is*

$$p_{2k}(x) = -x \prod_{j=1}^k (x^2 + (2j)^2)$$

*if  $m = 2k$  is even, and*

$$p_{2k+1}(x) = \prod_{j=0}^k (x^2 + (2j+1)^2)$$

if  $m = 2k + 1$  is odd. The eigenvalues of  $L_{2k}$  are  $0, \pm 2i, \pm 4i, \dots, \pm ki$ , while the eigenvalues of  $L_{2k+1}$  are  $\pm i, \pm 3i, \pm 5i, \dots, \pm(2k + 1)i$ .

We denote by  $\sigma(A)$  the spectrum of an operator  $A$ , and by  $S + T$ , resp.  $S \cdot T$ , the Minkowski sum  $\{s + t : s \in S, t \in T\}$ , resp. Minkowski product  $\{s \cdot t : s \in S, t \in T\}$ , of two sets  $S, T \subset \mathbb{C}$ . We write  $m \cdot S = \underbrace{S + \dots + S}_m$  and  $S^m = \underbrace{S \cdot \dots \cdot S}_m$ .

*Proof.* Recall that  $\gamma_m$  can be identified with the  $m$ -fold (symmetric) tensor product of  $\gamma_1$ . We have  $\gamma_1(s) = \exp(L_1 s)e_0 = \exp(Js)e_0$ , where  $e_0 = (1, 0)$  in  $\mathbb{R}^2$ . Therefore

$$\gamma_m(s) = (\exp(Js)e_0)^{\otimes m} = (\exp(Js))^{\otimes m} E_0 = \exp(L_m s) E_0.$$

For operators  $A, B$ ,  $\sigma(A \otimes B) = \sigma(A) \cdot \sigma(B)$ . Thus

$$\sigma(\exp(L_m s)) = \sigma(\exp(Js))^m = \{e^{is}, e^{-is}\}^m$$

and  $\sigma(L_m) = m \cdot \{\pm i\} = \{\pm(m - 2j)i : j = 0, 1, \dots, [\frac{m}{2}]\}$ , where  $[\cdot]$  denotes the greatest integer function. The formulas for the characteristic polynomials follow.  $\square$

We mention a related suggestive point of view. Let  $I_{m+1}$  denote the identity operator on  $\mathbb{C}^{m+1}$ . Define a linear operator  $\mathcal{L}_m$  by

$$(3.33) \quad \mathcal{L}_m = \frac{1}{\pi} \log((-1)^m I_{m+1}).$$

This suggestive notation means that  $\mathcal{L}_m$  is the diagonal matrix whose eigenvalues are  $\frac{1}{\pi}$  times  $m + 1$  particular values of  $\log((-1)^m)$ . When  $m = 2k$  these values are

$$mi, (m - 2)i, \dots, 2i, 0, -2i, \dots, -(m - 2)i, -mi$$

and when  $m = 2k + 1$  they are

$$mi, (m - 2)i, \dots, i, -i, \dots, -(m - 2)i, -mi.$$

The operators  $\mathcal{L}_m$  and  $L_m$  are similar over  $\mathbb{C}$  and hence have the same eigenvalues.

We observe explicitly part of Theorem 3.7. When  $m + 1$  is odd, one of the eigenvalues is zero. It follows in this case that  $\gamma_m$  maps to a hyperplane. For example, the image of  $\gamma_2$ , a priori in  $\mathbb{R}^3$ , is in fact a circle in the hyperplane  $\{(x_0, x_1, x_2) : x_0 + x_2 = 1\}$ , similarly, the image of  $\gamma_4$  lies in a four-dimensional hyperplane in  $\mathbb{R}^5$ .

**Remark 3.15.** We mention another role played by the curves  $\gamma_m$ . Using homogeneity and polar coordinates we extend  $\gamma_m$  to all of  $\mathbb{R}^2$  by setting

$$P_m(r \cos \theta, r \sin \theta) = r^m \gamma_m(\cos \theta, \sin \theta).$$

Then  $P_m$  defines a proper polynomial mapping from the unit disk to the unit ball in  $\mathbb{R}^{m+1}$ . This map  $P_m$  is invariant under a cyclic subgroup of order  $m$  of  $O(2)$ . See [9] and its references for information about holomorphic mappings invariant under finite subgroups of the unitary group. The particular group-invariant mappings  $z \mapsto z^{\otimes m}$  are the simplest examples, and their restrictions to the unit sphere in  $\mathbb{C}^m$  provide a complex variables analogue of the curves  $\gamma_m$ .

#### 4. CONCLUDING REMARKS

**Remark 4.1.** We discuss the issue of reparameterization. In this paper we have considered curves as maps from an interval in  $\mathbb{R}$  into  $\mathbb{R}^d$ . As previously mentioned, the class of curves in  $\mathcal{Q}_0$  is closed under affine reparameterization  $s \mapsto \lambda s + b$ . Under the correspondence in Theorem 3.9, such reparameterizations correspond to an equivalence of helical CR structures, where two such structures  $\mathcal{C}$  and  $\mathcal{C}'$  are called *equivalent* if the associated skew-symmetric matrices  $A$  and  $A'$  satisfy  $A' = \lambda A$  for some nonzero  $\lambda$ . Under the correspondence in Corollary 3.10, such reparameterizations correspond to isomorphic stratified Lie algebra structures. Similar statements can be made for curves in  $\mathcal{Q}_1$ ; we leave the precise statements to the reader.

**Remark 4.2.** The skew-symmetric matrices associated to the curves  $\gamma_m$  are bidiagonal, i.e., have non-zero entries only on the super- and sub-diagonals. Closely related issues arise in the fundamental algebraic question of normal forms for orthogonal similarity classes of skew-symmetric matrices. The paper [11] considers a skew-symmetric matrix  $A$  defined over an algebraically closed field  $\mathbb{F}$  and seeks an orthogonal  $P$  such that  $PAP^{-1}$  is as close to bidiagonal as possible. Achieving bidiagonality itself is not always possible. In our context the underlying skew-symmetric matrices are bidiagonal and real.

**Remark 4.3.** We can extend Corollary 3.10 to relate collections of curves in  $\mathcal{Q}_1$  to sub-Riemannian structures with additional vertical directions, enabling us to characterize arbitrary step two Carnot groups. For each  $\alpha = 1, \dots, p$ , let  $\mu_\alpha$  be a nonaffine curve in  $\mathcal{Q}_1$  which lies in a hyperplane in  $\mathbb{R}^d$ ; we assume that the horizontal spaces of these curves coincide. We further assume that the vertical directions  $w_\alpha$  for the curves  $\mu_\alpha$  are linearly independent in the orthogonal complement  $\mathbb{R}^p$ . A straightforward induction using Corollary 3.10 leads from this data to a unique step two stratified Lie algebra of type  $(2n, p)$  together with  $p$  normal geodesics  $c_1, \dots, c_p$  for the CC metric on the associated Lie group. Conversely, beginning from such a Lie algebra together with  $p$  geodesics, we construct  $p$  curves  $\mu_1, \dots, \mu_p$  in  $\mathcal{Q}_1$  taking values in  $\mathbb{R}^d$  with common horizontal space and vertical directions  $w_1, \dots, w_p$  forming a basis for  $\mathbb{R}^p$ . We obtain the following theorem.

**Theorem 4.4.** *Each  $p$ -tuple of nonaffine curves in  $\mathcal{Q}_1$ , each of which lies in a hyperplane in  $\mathbb{R}^d$ , with common horizontal space, and with vertical directions  $w_1, \dots, w_p$  forming a basis for  $\mathbb{R}^p$ , uniquely determines and is uniquely determined by a step two stratified Lie algebra  $\mathfrak{g}$  on  $\mathbb{R}^d$  together with a  $p$ -tuple of distinguished normal geodesics for the CC metric on the associated Carnot group.*

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